Musharakah Mutanaqisah Partnership
A Mathematical Derivation Note

P = Price of asset, e.g. a home
B_0 = Financier's contribution into the partnership
C_0 = Customer's contribution into the partnership

Therefore, $P = B_0 + C_0$

R = Periodic rental, eg. monthly
A = Additional periodic payment by customer to redeem the financier's equity faster

$M = R + A$, is therefore, the total periodic payment

Let C_i = the customer's equity (ownership) of the asset in period i

Let the proportion of customer's equity in period i, $r_i = \frac{C_i}{P}$

Therefore,

$$C_0 = C_0$$
$$C_1 = C_0 + r_0 R + A$$
$$C_2 = C_1 + r_1 R + A$$
$$C_3 = C_2 + r_2 R + A$$
.
.
.
$$C_n = C_{n-1} + r_{n-1} R + A$$

Therefore,

$$C_0 = C_0$$
$$C_1 = C_0 + r_0 R + A$$
$$C_2 = C_0 + r_0 R + A + r_1 R + A = C_0 + R(r_0 + r_1) + 2A$$
$$C_3 = C_0 + r_0 R + A + r_1 R + A + r_3 R + A = C_0 + R(r_0 + r_1 + r_2) + 3A$$
.
.
.
$$C_n = C_0 + R(r_0 + r_1 + r_2 + ... + r_{n-1}) + nA$$
$$C_n = C_0 + \frac{R}{P}(C_0 + C_1 + C_2 + ... + C_{n-1}) + nA \qquad \text{Since } r_i = \frac{C_i}{P}$$

Let $x = \frac{R}{P}$, then

$$C_1 = C_0 + xC_0 + A = (1+x)C_0 + A$$

$$C_2 = C_0 + x\left(C_0 + C_0 + xC_0 + A\right) + 2A = (1 + 2x + x^2)C_0 + (x+2)A$$

$$\begin{aligned} C_3 &= C_0 + x\left(C_0 + C_0 + xC_0 + A + C_0 + \left(C_0 + C_0 + xC_0 + A\right) + 2A\right) + 3A \\ &= (1 + 3x + 3x^2 + x^3)C_0 + (x^2 + 3x + 3)A \end{aligned}$$

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Therefore,

$$C_1 = (1+x)C_0 + A$$

$$C_2 = (1+x)^2 C_0 + (x+2)A$$

$$C_3 = (1+x)^3 C_0 + (x^2 + 3x + 3)A$$

$$C_4 = (1+x)^4 C_0 + (x^3 + 4x^2 + 6x + 4)A$$

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$$C_n = (1+x)^n C_0 + \left[\frac{(1+x)^n - 1}{x}\right]A \qquad (1)$$

and, of course, the proportion of the customer's equity in the n^{th} period is $r_n = \frac{C_n}{P}$

Rewriting equation (1), the number of periods taken by the customer to fully own the house is given by, where $C_n = P$,

$$\begin{aligned} P &= (1+x)^n C_0 + \frac{(1+x)^n}{x}A - \frac{1}{x}A \\ &= (1+x)^n\left[C_0 + \frac{A}{x}\right] - \frac{1}{x}A \end{aligned}$$

$$(1+x)^n = \frac{P + \frac{A}{x}}{C_0 + \frac{A}{x}}$$

$$\Rightarrow n = \frac{\ln\left(P + \frac{A}{x}\right) - \ln\left(C_0 + \frac{A}{x}\right)}{\ln(1+x)} \tag{2}$$

Once the rental, *R*, has been determined and the customer has decided on the period of partnership, i.e. the *n*, then the periodic amount the customer has to top up additionally is given by

$$A = \frac{x\left[P - (1+x)^n C_0\right]}{(1+x)^n - 1} \tag{3}$$

and, the formula for determining the periodic payment is given by

$$\begin{aligned} M &= R + A \\ &= \frac{R\left[(1+x)^n - 1\right] + x\left[P - (1+x)^n C_0\right]}{(1+x)^n - 1} \\ &= \frac{R(1+x)^n - R + xP - x(1+x)^n C_0}{(1+x)^n - 1} \\ &= \frac{x(1+x)^n P - R + R - x(1+x)^n C_0}{(1+x)^n - 1} \quad \text{Since} \quad xP = R \\ &= \frac{x(1+x)^n \left[P - C_0\right]}{(1+x)^n - 1} \\ &= \frac{x(1+x)^n B_0}{(1+x)^n - 1} \end{aligned}$$

$$\Rightarrow M = \frac{x(1+x)^n B_0}{(1+x)^n - 1} \tag{4}$$

which, interestingly, is similar to the normal annuity formula used for computing the payment in conventional loan calculations. **Hence, mathematically, the normal annuity formula can also be used for *Mushārakah Mutanākisah* calculations, but the periodic interest rate is replaced by the rental rate, $\boldsymbol{x = \frac{R}{P}}$.** Indeed then, the periodic rate of return for *Mushārakah Mutanākisah* Partnership is solely determined by the rental rate, $x = \frac{R}{P}$. Therefore, the

$$\text{Internal Rate of Return (IRR) to bank} = \frac{R}{P} \tag{5}$$

Also,

Total payment made to financier = Mn (6)

Total profit to financier = $Mn - B_0$ (7)

Note: Since the rate of return (IRR) for the financier is solely determined by the rental rate, $x = \frac{R}{P}$, **irrespective** of the initial capital provided by the financier (B_0) and/or the duration of the partnership (n), the financier may be tempt **to finance only homes with high rental values**; while it is in the interest of the customers to negotiate for low rentals. At the extreme, if the rental is nil, then the *Mushārakah Mutanākisah* financing will become similar to *Qard al-Hassan*.

Worked Example

Price of House = RM200,000
Rent = RM1,000 per month

Conventional Mortgage & BBA	Musharakah Mutanaqisah
APR = 10% Downpayment = RM20,000 (10%) Loan OR Financing = RM180,000 (90%) Duration = 20 Years monthly payments In conventional mortgage the Present Value of Annuities formula is used: $PV = \frac{Pmt}{i}\left[1 - \frac{1}{(1+i)^n}\right]$ The monthly interest rate, $i = \frac{10\%}{12} = 0.8333\% = 0.008333$ i.e. $180{,}000 = \frac{Pmt}{0.008333}\left[1 - \frac{1}{(1.008333)^{240}}\right]$ $\Rightarrow Pmt = RM1{,}737.03$ per month **Using Finance Calculator:** **PV = -180,000 i = 0.8333% n = 240 Pmt = ?** Total payments = RM1,737.03 X 240 = RM416,889.35 Total interest = RM416,889.35 – 180,000 = RM236,889.35 (Which is total profit under BBA) Balance after 10 years (120 payments) **Under Conventional** $Balance = \frac{1{,}737.03}{0.008333}\left[1 - \frac{1}{(1.008333)^{120}}\right]$ $= RM131{,}443.76$	Initial Contribution of customer = RM20,000 (10%) Initial Contribution of financier = RM180,000 (90%) Rental rate, $\frac{x}{P} = \frac{1{,}000}{200{,}000} = 0.5\%$ per month Duration = 20 Years monthly payments In order to redeem the financier's share within 20 years, the customer has to pay additional amount, A, per month over and above the rental [Equation 3] $A = \frac{x\left[P - (1+x)^n C_0\right]}{(1+x)^n - 1}$ $= \frac{0.005\left[200{,}000 - (1.005)^{240} \times 20{,}000\right]}{(1.005)^{240} - 1}$ $= RM289.58$ Total monthly payment = RM1000 + RM289.58 = RM1,289.58 Since the mathematical derivation showed that the formula for MM is similar to those of conventional and BBA but interest rate replaced with rental rate, we can also solve the above as follows: $180{,}000 = \frac{Pmt}{0.005}\left[1 - \frac{1}{(1.005)^{240}}\right]$ $\Rightarrow Pmt = RM1{,}289.58$ per month **Using Finance Calculator:** **PV = -180,000 i = 0.5% n = 240 Pmt = ?** Since the montly rent is RM1,000, then the additional amount needed is, therefore, RM289.58 Balance after 10 years (240 payments), i.e. the financier's equity in the house **Under MM**

Using Finance Calculator:
i = 0.8333% Pmt = 1,737.03 n = 120 PV = ?

Under BBA
The balance is simply the monthly payment times the number of months

Balance = RM1,737.03 X 120
= RM208,444.66

$$Balance = \frac{1{,}289.58}{0.005}\left[1-\frac{1}{(1.005)^{120}}\right]$$

$$= RM116{,}156.56$$

Using Finance Calculator:
i = 0.5% Pmt = 1,289.58 n = 120 PV = ?

If customer wants to own the home in 15 years, i.e. 180 months, then

$$180{,}000 = \frac{Pmt}{0.005}\left[1-\frac{1}{(1.005)^{180}}\right]$$

$$\Rightarrow Pmt = RM1{,}518.94 \text{ per month}$$

i.e. an additional monthly amount of RM518.94 is needed.

Using Finance Calculator:
PV = -180,000 i = 0.5% n = 180 Pmt = ?

If the customer pays RM1,737.03 as in the BBA example, hw could own the house even faster, i.e. [Equation 2]

$$n = \frac{\ln\left(P+\frac{A}{x}\right)-\ln\left(C_0+\frac{A}{x}\right)}{\ln(1+x)}$$

$$= \frac{\ln(200{,}000+147{,}406)-\ln(20{,}000+147{,}406)}{\ln(1.005)}$$

$$= 146.38 = 147 \text{ months}$$

i.e. 12 years 3 months.

Using Finance Calculator:
PV = -180,000 i = 0.5% Pmt = 1,737.03 n = ?

Disclaimer: The above mathematical derivation and examples are meant for educational purposes and not meant for investment advice or otherwise.

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